APPENDIX 27

REAL OPTIONS METHODOLOGY

APPENDIX OUTLINE

In the main body of Chapter 27 in the printed text we presented the basic principles and general results of option valuation theory as applied to speculative land valuation and basic development project evaluation and decision making. In this appendix, we will extend sections 27.4 and 27.5 of the main text to present the details of what you need to know to actually apply the real options model to specific land valuation and simple development project problems. (Applications to more complex development projects are discussed in Chapter 29.) Sections 27A.1 and 27A.2 in this appendix are extensions of the material in the printed text sections 27.4 and 27.5, respectively, and they are complemented by an Excel file with templates and examples of the applications presented here.

27A.1 The Binomial Model of Option Value

The one-period binomial world in which the development option examples of sections 27.2 and 27.3 were presented is obviously a tremendous simplification of reality. In the real world, time is continuous, and asset values can assume many more than just two possible outcomes. But the binomial model is more than just a pedagogical simplification device. It is a building block that can be used to construct a much more general and realistic option valuation and analysis tool. This is because the dynamics of the evolution of the value of built property through time, upon which the land value and the optimal development policy depend, can be modeled as a series of single-period binomial outcome possibilities. Within each period, we can apply the tools presented in the preceding section. By stitching the individual binomial periods together sequentially, we can span as long a time frame as we like, and by making each individual binomial period as short as we want, we can get the model to realistically approach continuous time and continuous pricing. In this section, we will see how this works in detail.

27A.1.1 Building the Underlying Asset Value "Tree"

To begin, let's analyze in a bit more depth how our previous example binomial world was constructed, so that we can see how to extend this model to represent a more realistic and general world. Recall that a new office building today can be observed to be worth $100 million (if it already exists). Next year, that building (or to be more precise, a similar new one then) will be worth either $113.21 million with 70 percent probability, or 78.62 million with 30 percent probability. Where did we get these numbers?

In fact, these numbers were selected so that the binomial world we created would converge, as the length of time in each period became shorter and the number of sequential binomial periods became greater, toward a continuous time world that has the following characteristics for the dynamics of office building value:

- **9%** $= \boldsymbol{r_V} =$ *Expected annual total return on investment in completed office buildings.*
- **6%** $= \boldsymbol{y_V} =$ *Annual net rental income cash payout (yield) as a fraction of current building value.*
- **20%** $= \boldsymbol{\sigma_V} =$ *Expected annual volatility of returns on* individual *completed office buildings.*[1]

Note that the above assumptions regarding rV and yV carry the implication that the expected annual growth rate in (new) office building values is:

$$\begin{aligned} g_V &= (1 + r_V)/(1 + y_V) - 1 \\ &= 1.09/1.06 - 1 = 2.83\%. \end{aligned} \tag{A.1}$$

One can start out specifying any two of the three return component variables for the underlying asset, r_V, y_V, or g_V, and the third will be automatically determined by the relationship in equation (A.1).[2]

We can model the above dynamics in a binomial world as follows. Letting V_0 be the current (observable) value of the underlying asset (the $100 million in our previous example), the next year binomial outcome values are set at:[3]

$$\begin{aligned} V_1^{up} &= V_0 \times (1 + \sigma_V)/(1 + y_V) \\ V_1^{down} &= V_0/(1 + \sigma_V)/(1 + y_V) \end{aligned} \tag{A.2a}$$

And the probability of the *up* outcome is set at:[4]

$$p = ((1 + r_V) - 1/(1 + \sigma_V))/((1 + \sigma_V) - 1/(1 + \sigma_V)) \tag{A.2b}$$

(The probability of the down outcome is of course just 1 − p.) Thus, in the case of our example:

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[1]Note that the relevant volatility to use in the context of an option to develop a single asset is the single-asset volatility, which includes the idiosyncratic risk of the individual building, and thus is larger than the volatility that would typically be measured for a real estate market index or a portfolio of many properties. (See discussions about real estate volatility in Chapters 9 and 25.)

[2]With this in mind, it will often be advisable to make the estimates of g_V and y_V as realistic as possible (letting r_V follow from those estimates), because it turns out (as we shall see later) that the results of the option model are independent of the value chosen for r_V. In this regard, note that the growth rate measured by g_V is that in the value of *new* buildings in the market, not reflecting the depreciation in existing structures. (Recall our discussions in Chapters 5 and 11 regarding the real depreciation of existing structures.)

[3]Recalling Appendix 10C, note therefore that:

$$V_1^{up}\% - \frac{V_1^{up}}{PV[V_1]} - \frac{V_0(1+\sigma_V)/(1+y_V)}{E_0[V_1]/(1+r_V)} - \frac{V_0(1+\sigma_V)/(1+y_V)}{V_0(1+g_V)/(1+r_V)} - \frac{V_0(1+\sigma_V)/(1+y_V)}{V_0/(1+y_V)} - 1+\sigma_V$$

and :

$$V_1^{down}\% - \frac{V_1^{down}}{PV[V_1]} - \frac{V_0/(1+\sigma_V)/(1+y_V)}{E_0[V_1]/(1+r_V)} - \frac{V_0/(1+\sigma_V)/(1+y_V)}{V_0(1+g_V)/(1+r_V)} - \frac{V_0/(1+\sigma_V)/(1+y_V)}{V_0/(1+y_V)} - 1/(1+\sigma_V)$$

[4]There are alternative recipes for setting up the binomial value tree for the underlying asset. In essence, we have three parameters to determine: the *up* ratio, the *down* ratio, and the *up* probability. And we have two parameters describing the underlying asset value probability distribution that we wish to match: the mean and the volatility (standard deviation). Solving two equations for three unknowns can be done in more than one way. The approach described here, which fixes the *up* ratio to be the inverse of the *down* ratio, is called the "CRR" method, after its originators: Cox, Ross, and Rubinstein (*JFE* 1979). An alternative approach to the one described here is to fix the probability at 50%. Then the *up* ratio will be $(1 + r_V)$ $(1+r_V)(1+\sigma_V)/(1+\sigma_V^2/2)$ instead of just $(1 + \sigma_V)$, and the *down* ratio will be $(1+r_V)(1+\sigma_V)/(1+\sigma_V^2/2)$. However, only the CRR approach is consistent with a rigorous economic valuation model for the option in discrete time (the arbitrage or equilibrium valuation described in section 27.3). The disadvantage of the CRR approach is that the standard deviation of the underlying asset return distribution does not exactly equal the input instantaneous rate of σ_V except in the continuous time limit. All the approaches converge to the same thing as period length approaches zero (continuous time), and all are useful mathematical approximations even in discrete time.

$$
\begin{aligned}
V_1^{up} &= \$100 \times 1.20/1.06 = \$113.21 \text{ million;} \\
V_1^{down} &= \$100/1.20/1.06 = \$78.62 \text{ million.} \\
p &= (1.09 - 0.83)/(1.20 - 0.83) = 0.26/0.37 = 70\%.
\end{aligned}
$$

Now let's shorten the length of the time periods we are modeling, from one year, to one month. Supposing that our previously given return and volatility parameters represent nominal annual rates, to shorten the periods to months we divide the given nominal annual rates by 12 for the return components, and by the square root of 12 for the volatility. We still use formulas (A.1) and (A.2) in the model, only now we have for each period (month): $r_V = 9\%/12 = 0.75\%$; $y_V = 6\%/12 = 0.50\%$, $g_V = (1.0075/1.0050) - 1 = 0.25\%$, and $\sigma_V = 20\%/\sqrt{12} = 20\%/3.46 = 5.77\%$.[5] Thus, applying formula (A.1), we have:

$$
\begin{aligned}
V_1^{up} &= \$100 \times 1.0577/1.005 = \$105.25 \text{ million;} \\
V_1^{down} &= \$100/1.0577/1.005 = \$94.07 \text{ million.} \\
p &= (1.0075 - 0.9454)/(1.0577 - 0.9454) = 0.062/0.112 = 55.27\%.^{6}
\end{aligned}
$$

The *up* outcome is now seen to be \$105.25 million with 55.27 percent probability, and the *down* outcome is \$94.07 million with 44.73 percent probability, *next month*.

From each of these two future possible *"states of the world,"* two further future states of the world are possible the month after that. Thus, we can extend our binomial projection to that second month (two months from the present), assuming that the same one-period dynamics apply to the underlying asset in every period. From next month's "*up*" outcome, the possible month 2 outcomes will be:

$$
\begin{aligned}
V_{0,2} &= \$105.25 \times 1.0577/1.005 = \$110.77 \text{ million;} \\
V_{1,2} &= \$105.25/1.0577/1.005 = \$99.01 \text{ million.}
\end{aligned}
$$

From next month's *"down"* outcome, the two further possibilities in month 2 are:

$$
\begin{aligned}
V_{1,2} &= \$94.07 \times 1.0577/1.005 = \$99.01 \text{ million;} \\
V_{2,2} &= \$94.07/1.0577/1.005 = \$88.49 \text{ million.}
\end{aligned}
$$

(Notice that we are adopting a labeling convention in which the asset is given subscripts "*i*" and "*j*" that characterize the state of the world, with "*i*" representing the total number of *down* outcomes since the present, and "*j*" representing the total number of periods of time since the present.) These results are presented graphically in Exhibit 27A-1.

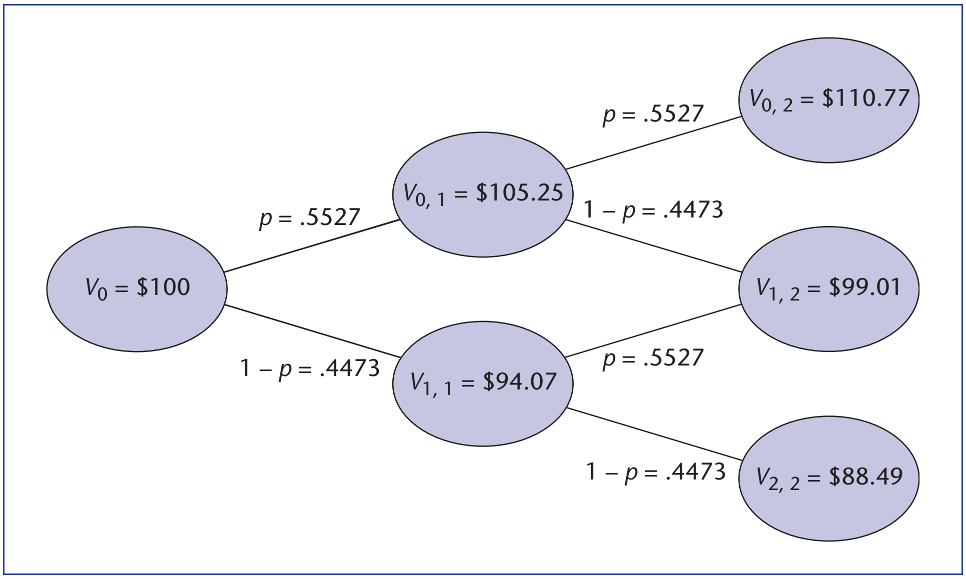
Notice a remarkable (and very important) feature of the above procedure: the *down* outcome from next month's *up* scenario is *the same value* as the *up* outcome from next month's *down* scenario, namely, \$99.01 million. This is no accident. Mechanically, it results simply from the commutative property of multiplication and our assumption of constant volatility and payout rate of the underlying asset.[7] From any current underlying asset value $V_{i,j}$, the

[5]True nerds tend to use input parameters quoted as "instantaneous rates" (the "continuously compounded returns" or log-differences described in section 8.1.6 of Chapter 8). In that case, supposing the option has a lifetime of T years and for the binomial tree we divide the time until then into n equal-length discrete periods, we have the *up* ratio is $V_1^{up}/PV[V_1] - \exp[\sigma_V \sqrt{T/n}]$, the *down* ratio is $V_1^{down}/PV[V_1] - \exp[-\sigma_V \sqrt{T/n}]$, and the probability is $p - (\exp[r_V T/n] - \exp[-\sigma_V \sqrt{T/n}])/(\exp[\sigma_V \sqrt{T/n}] - \exp[-\sigma_V \sqrt{T/n}])$.

[6]As noted elsewhere in this text, answers in numerical examples may appear inconsistent because the answers are derived from more decimal precision than is shown in intermediate calculation steps.

[7]Note that an assumption of a constant expected return (or growth rate) in the underlying asset is not technically required in this option valuation model. We can modify the growth or return expectations by modifying the assumed values of g_V or r_V in any future state of the world (holding y_V constant). For example, we could make the expected return (and growth rate) be a function of the time period or of the value of the underlying asset, or of both (so as to represent a cyclical or "mean reverting" market for the underlying asset, for example). Constant growth is, however, the classical assumption (and the simplest baseline case). It corresponds to an asset market that is sufficiently efficient so as to have "memoryless" prices (i.e., a market in which future returns are not predictable based on past returns, because current asset prices reflect all relevant currently available information).

EXHIBIT 27A-1 Extending the Binomial Tree to a Second Month



up outcome is $V_{i,j+1} = V_{i,j}/(1 + y_V)$ *times* $(1 + \sigma_V)$ and the *down* outcome is $V_{i+1,j+1} = V_{i,j}/(1 + y_V)$ *divided by* $(1 + \sigma_V)$, the multiplicative inverse of each other. Starting from any $V_{i,j}$ value, two periods later unless there have been two *up* moves in a row or two *down* moves in a row, the outcome will be either *up times down* or *down times up*, which in either case because of commutativity and inverse cancellation ends up at $V_{i+1,j+2} = V_{i,j}/(1 + y_V)^2$.

From a computational perspective, these mechanics are vital, as they cause the future possible values of the underlying asset (the future states of the world) to "recombine" at intermediate points, which greatly reduces the number of possible future states of the world that must be considered.[8] More importantly from an economic perspective, these mechanics cause the model to converge toward a "normal" (Gaussian) distribution for the underlying asset returns as the time periods become shorter (assuming constant volatility).

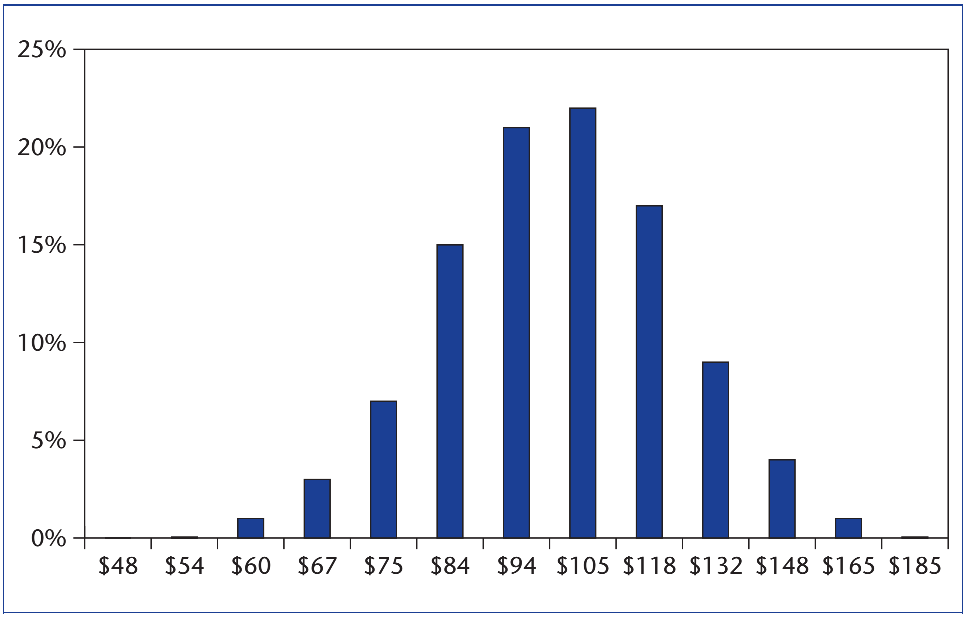
Note also that, while the binomial probability of the single-period *up* and *down* outcomes from either of the two possible month 1 values of the new office building remain 55.27 percent and 44.73 percent, respectively, from the standpoint of the present ("month 0"), the probabilities of the *three* possible future values of the underlying asset two months from now are:

$$\text{Prob}[V_{0,2} = \$110.77] = (0.5527)(0.5527) = 30.55\%;$$
$$\text{Prob}[V_{1,2} = \$99.01] = (0.5527)(0.4473) + (0.4473)(0.5527) = 49.44\%;$$
$$\text{Prob}[V_{2,2} = \$110.77] = (0.4473)(0.4473) = 20.00\%.$$

Of course, these three probabilities sum to 100 percent, and the larger middle probability reflects the fact that there are two "routes" by which the value can end up at \$99.01 million in two periods from \$100 million today (either *up* followed by *down* or *down* followed by *up*) while there is only one "route" by which the other two values can be obtained (either two *ups* or two *downs*, respectively).

[8]Without this recombination property, the number of future possible states of the world that we would have to consider n periods in the future would be 2^n. With recombination, the number of future states simply equals $n + 1$. Thus, to consider 12 monthly periods into the future, with recombination we have 13 states of the world at the end of the year, and without recombination we would have 4,096 states of the world

EXHIBIT 27A-2 One Year (Period 12) Value Probabilities for the Underlying Asset (new office building)



Suppose we continue building the underlying asset binomial "tree" forward in time out more and more periods in the same manner as we just did, out to a total of 12 periods (One year) from the present. And suppose we enumerate all of the resulting possible underlying asset value outcomes one year in the future and tabulate the probability, as of the present, of each of those possible outcomes (the "unconditional" probabilities) in the same manner as above (i.e., multiplying the probabilities across steps and counting up the number of "routes" to each outcome). Then, the probability distribution of the future possible values of the office building one year from now will be modeled as shown graphically in Exhibit 27A-2. Notice that this is the type of "bell shaped" probability distribution that we would realistically expect.[9] The mean of this value distribution is $E_0[V_{12}] = V_0((1 + r_V) = (1 + y_V))^{12} = V_0(1 + g_V)^{12} = \$100 \times (1.0075/1.005)^{12} = \$100 \times 1.0025^{12} = \$103$ million, and its standard deviation as a fraction of the starting value of $100 million is 20 percent, thus reflecting the growth dynamics parameters (mean and volatility) that we input into the model. This demonstrates how, even with as few as 12 periods per year, the binomial model can well represent a realistic model of underlying asset value dynamics, in terms of mean, volatility, and general shape of the probability distribution.

27A.1.2 Evaluating the Option

The table in Exhibit 27A-3 shows the complete underlying asset value tree for the 12-month option to develop the office building. (Note that the first two columns reflect the numbers we have previously calculated.[10]) Each *i,j* cell in the table represents a future possible state of the world between now and next year.

[9]To be more precise, the binomial model with constant dynamics as we have described here will converge toward a normally distributed (symmetric) continuously compounded return distribution, which equates to a log-normal distribution of possible future asset values (which is slightly skewed to the right, as negative values of the underlying asset are not possible).

[10]This and all option valuation examples and tools described in this chapter are available as Excel templates included on the CD accompanying this book. Note that in this table, reading horizontally from left to right across columns within a row represent *up* outcomes, while moving down one row from the previous column represents down outcomes.

Office building values ("*V*" ex-dividend, new):

Month ("*j*"): "down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	*n* = 12
0	100.00	105.25	110.77	116.58	122.70	129.14	135.91	143.05	150.55	158.45	166.77	175.52	184.73
1		94.07	99.01	104.20	109.67	115.43	121.48	127.86	134.57	141.63	149.06	156.88	165.11
2			88.49	93.14	98.02	103.17	108.58	114.28	120.28	126.59	133.23	140.22	147.58
3				83.25	87.62	92.21	97.05	102.14	107.50	113.15	119.08	125.33	131.91
4					78.31	82.42	86.75	91.30	96.09	101.13	106.44	112.02	117.90
5						73.67	77.53	81.60	85.89	90.39	95.13	100.13	105.38
6							69.30	72.94	76.77	80.79	85.03	89.49	94.19
7								65.19	68.61	72.21	76.00	79.99	84.19
8									61.33	64.55	67.93	71.50	75.25
9										57.69	60.72	63.91	67.26
10											54.27	57.12	60.12
11												51.05	53.73
12													48.03

Previously Calculated

EXHIBIT 27A-3 Underlying Asset Value Tree for 12 Months

To evaluate the option, the next step is to identify construction cost values corresponding to each future time period, reflecting the expected growth in construction costs. The table in Exhibit 27A-4 presents such values corresponding to each state of the world represented previously in Exhibit 27A-3, reflecting our previous assumption of an expected construction cost growth rate of 2 percent per year, starting from $88.24 million today and growing to $90 million next year.[11] Consistent with our previous assumption of riskless construction costs (recall that construction of the office building was going to cost $90 million no matter which scenario occurred), all the cost figures in Exhibit 27A-4 are the same within each column (i.e., there is no uncertainty about future construction costs).[12]

The final step is to compute the option values corresponding to each state of the world. This is done using the same certainty-equivalence DCF technique we described in section 27.3.2, applied one cell at a time, starting from the terminal period (in this case, month 12), working backwards in time from right to left across the table. (Note that while the underlying value tree is built form left to right working forward in time, the option value tree based on that underlying value tree is built from right to left working backward in time.)

Thus, we begin by computing the values the option will have in each of the 13 states of the world that are possible for month 12. For example, in state 0,12 the new office building would be worth $184.73 million, and in state 1,12 it would be worth $165.11 million (see column 12 in Exhibit 27A-3). In either case, the construction cost is $90 million. Thus, the NPV of immediate development (recall our instantaneous construction assumption) would be $184.73 − $90 = $94.73 million in state 0,12, or $165.11 − $90 = $75.11 million in state 1,12.

[11]The 2% rate is a simple annual rate. The equivalent nominal annual rate with monthly compounding is 1.98%, implying a simple monthly rate of: $(90.00/88.24)^{1/12} - 1 = 0.1652\%$.

[12]This is a simplifying assumption that might reflect, for example, the use of fixed-price construction contracts. Risky construction costs can be incorporated into the model by a mathematical transformation, computing the option value *per dollar of construction cost*. This transformation is discussed in section 27A.2 in the context of the Samuelson-McKean formula.

Office Building Construction Costs ("*K*"):

Month ("*j*"): "down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	$n = 12$
0	88.24	88.38	88.53	88.67	88.82	88.97	89.11	89.26	89.41	89.56	89.70	89.85	90.00
1		88.38	88.53	88.67	88.82	88.97	89.11	89.26	89.41	89.56	89.70	89.85	90.00
2			88.53	88.67	88.82	88.97	89.11	89.26	89.41	89.56	89.70	89.85	90.00
3				88.67	88.82	88.97	89.11	89.26	89.41	89.56	89.70	89.85	90.00
4					88.82	88.97	89.11	89.26	89.41	89.56	89.70	89.85	90.00
5						88.97	89.11	89.26	89.41	89.56	89.70	89.85	90.00
6							89.11	89.26	89.41	89.56	89.70	89.85	90.00
7								89.26	89.41	89.56	89.70	89.85	90.00
8									89.41	89.56	89.70	89.85	90.00
9										89.56	89.70	89.85	90.00
10											89.70	89.85	90.00
11												89.85	90.00
12													90.00

EXHIBIT 27A-4 Construction Cost Tree for 12 Months (in $ millions)

As the option expires after period 12, its values in all the other period 12 states can be computed similarly as:

$$C_{i,12} = MAX\ [V_{i,12} - 90,\ 0]$$

In this way, all of column 12 in the option value tree is filled in.

To demonstrate the remainder of the option valuation process, let us begin for illustrative purposes by computing the value of a European option, expiring in one year, to develop the office building. That is, suppose the office building can only be developed in month 12, not before. Consider the value of this option one month prior to its expiration, in the highest possible state of the world at that time, state 0,11. In this state of the world, the only possible future value outcomes for the option are either the 0,12 state value of $94.73 million or the 1,12 state value of $75.11 million that we previously computed. The value of the option is therefore computed using the certainty-equivalence formula (1) presented in section 27.3:

$$C_{0,11} = \frac{E[C_{12}] - (C_{0,12}\$ - C_{1,12}\$)\dfrac{E[r_V] - r_f}{V_{0,12}\% - V_{1,12}\%}}{1 + r_f}$$

$$= \frac{((.5527)\$94.73 + (.4473)\$75.11) - (\$94.73 - \$75.11)\dfrac{.0075 - .0025}{1.0577 - 1/1.0577}}{1.0025}$$

$$= \frac{\$85.95 - (\$19.62)\dfrac{.005}{.1123}}{1.0025} = \frac{\$85.08}{1.0025} = \$84.87$$

In this way, we compute the value of the option in each state of the world in period (column) 11, in each case based on the two possible option values the following period. Once we have computed the option values in all of the period 11 states, we can move back to period (column) 10 and compute all the option values in that period in the same manner. We can thus

Development is possible only in month 12:

Month ("j"): "down" moves ("i"):	0	1	2	3	4	5	6	7	8	9	10	11	n = 12
0	11.09	14.67	19.01	24.10	29.89	36.31	43.25	50.64	58.47	66.77	75.56	84.87	94.73
1		7.45	10.27	13.84	18.23	23.41	29.32	35.82	42.80	50.19	58.03	66.32	75.11
2			4.58	6.63	9.38	12.96	17.43	22.75	28.79	35.38	42.36	49.75	57.58
3				2.48	3.81	5.73	8.42	12.02	16.61	22.14	28.35	34.93	41.91
4					1.12	1.85	2.99	4.74	7.33	10.99	15.83	21.69	27.90
5						0.38	0.67	1.19	2.09	3.59	6.04	9.85	15.38
6							0.07	0.14	0.28	0.55	1.08	2.12	4.19
7								0.00	0.00	0.00	0.00	0.00	0.00
8									0.00	0.00	0.00	0.00	0.00
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

EXHIBIT 27A-5 European Option Value Tree for 12-Month Option

work all the way back to value the option in the present (time 0). As seen in Exhibit 27A-5, the result is a present value of $11.09 million for the option to develop in month 12 (but not before).

The valuation of the American option (that allows development in *any* month prior to expiration of the option) proceeds in the same way, only at each node the value of the option is the *maximum* of either its immediate exercise value or the present value of holding the option one more period. The latter value is computed in the same way as with the European option, using the certainty-equivalence DCF formula. Thus, the complete formula for the valuation of the American option in the binomial world comes in two parts, embedded in an algorithm. The first step in the algorithm is to compute the value of the option in the period of its expiration, which is the same as for the European option, as given in (A.3a) below. Supposing option expiration in period $j = T$:

$$C_{i,T} = MAX\ [V_{i,T} - K_T, 0]. \tag{A.3a}$$

Next, the option value is computed in each state of the world i, j prior to the terminal period working backwards in time (first for all i in $j = T - 1$, then for all i in $j = T - 2$, and so on, back finally to $j = 0$). The option valuation formula in each state i, j for $j < T$ (where i is the number of *up* outcomes since the present and j is the total number of periods since the present) is given in formula (A.3b) below.

$$\begin{aligned} C_{i,j} &= MAX\left[V_{i,j} - K_j, \frac{E[C_{j+1}] - (C_{i,j+1}\$ - C_{i+1,j+1}\$)\dfrac{E[r_V] - r_f}{V_{i,j+1}\% - V_{i+1,j+1}\%}}{1 + r_f}\right] \\ &= MAX\left[V_{i,j} - K_j, \frac{(pC_{i,j+1} + (1-p)C_{i+1,j+1}) - (C_{i,j+1}\$ - C_{i+1,j+1}\$)\dfrac{E[r_V] - r_f}{(1+\sigma_V) - 1/(1+\sigma_V)}}{1 + r_f}\right] \end{aligned} \tag{A.3b}$$

where p is as defined in formula (A.2b).

Development possible in any month:

Month ("*j*"): "down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	*n* = 12
0	12.57	16.87	22.24	27.91	33.88	40.17	46.80	53.79	61.14	68.90	77.06	85.67	94.73
1		8.19	11.41	15.58	20.85	26.46	32.37	38.60	45.16	52.07	59.35	67.03	75.11
2			4.91	7.15	10.22	14.28	19.47	25.02	30.87	37.03	43.53	50.37	57.58
3				2.61	4.02	6.08	8.99	12.94	18.10	23.59	29.38	35.48	41.91
4					1.16	1.91	3.11	4.95	7.68	11.57	16.73	22.17	27.90
5						0.38	0.69	1.22	2.14	3.70	6.25	10.28	15.38
6							0.07	0.14	0.28	0.55	1.08	2.12	4.19
7								0.00	0.00	0.00	0.00	0.00	0.00
8									0.00	0.00	0.00	0.00	0.00
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

EXHIBIT 27A-6 American Option Value Tree for 12-Month Option

The result in our numerical example is that the present value of the option is $12.57 million, as seen in Exhibit 27A-6.[13]

Note also that with an American option, it is possible to identify states of the world in which immediate exercise of the option is optimal, and those in which it is optimal to continue holding the option. In the real options model of land, these would correspond to states of the world in which immediate development is triggered, and those in which the land is continued to be held for speculation. Exhibit 27A-7 shows the optimal exercise binomial tree for our example 12-month option. Note that it is optimal at present to continue holding the land for speculation, and that the first state in which development would be optimal would occur two months from now, if and only if there are two *up* outcomes in the value of office buildings between now and then (state 0,2). In that case, the value of a new office building would at that time be observed to have grown to $110.77 million from today's $100 million value.

Note that in the binomial optimal exercise tree there is in general a *boundary line* above and to the right of which it is optimal to exercise, and below and to the left of which it is optimal to continue holding the option. This boundary line can be defined by a ratio of current $V_{i,j}/K_j$ values that must be sufficiently high in order to trigger optimal immediate development. In other words, the construction benefit cost ratio must be high enough to make it worthwhile to give up the option premium, the potential value of

[13]Note that the modeled option value tends to "oscillate" as a function of whether the option's lifetime is modeled using an even or odd number of periods. For example, if we modeled our example one-year option using 11 equal-length periods (each one 12/11 months long), we would get a valuation of $12.51 million instead of $12.57 million. However, if we modeled the one-year option using 10 periods (each 1/10th of a year long), then we would get an option valuation of $12.58 million. As we previously saw in section 12.3, with only one period (of one-year length), the option was valued at $12.09 million. The exact relationship between the period length and the convergence to the continuous time valuation is complex and depends upon the parameters of the problem.

Development is possible only in month 12:

Month ("*j*"): "down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	*n* = 12
0	hold	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		hold	hold	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			hold	hold	hold	hold	exer	exer	exer	exer	exer	exer	exer
3				hold	hold	hold	hold	hold	exer	exer	exer	exer	exer
4					hold	hold	hold	hold	hold	exer	exer	exer	exer
5						hold	hold	hold	hold	hold	hold	exer	exer
6							hold	hold	hold	hold	hold	hold	exer
7								hold	hold	hold	hold	hold	hold
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

Optimal Exercise Boundary

EXHIBIT 27A-7 American Option Optimal Exercise for 12-Month Option

waiting. In general, this *"hurdle ratio"* will be higher the greater the volatility (σ_V) and the lower the cash payout rate (y_V) in the underlying asset, the lower the growth rate in the exercise price (g_K), and the longer the time until expiration of the option. As option expiration approaches, the value of the option premium diminishes toward zero, and exercise becomes optimal as the current value of the underlying asset just barely exceeds its current construction cost.

27A.1.3 Accounting for Construction Time

In all of the option valuation examples we have considered so far, we have assumed that construction was instantaneous, resulting in the immediate completion of a fully operating building as soon as the development decision is made. This, of course, is unrealistic, and to make our development option model realistic, we need to account for the time it takes to complete construction. This is often referred to as the **time to build**. Under the assumption that the construction time is known in advance (or is a deterministic function of the time period or of the construction cost, or of any variable that is determined in the binomial tree), it is very straightforward to account for time to build in the option valuation model.

The effect of time to build is that when the decision to exercise the development option is made (the decision to begin construction), instead of immediately obtaining a completed fully operating building, one obtains a *forward claim* on a fully operating building. A "forward claim" is an enforceable contract that will give its holder the claimed asset at a definite future point in time. For example, suppose the option is exercised in month t, and the time to build is 18 months. Then, upon exercise at time t we obtain, for certain, a fully operating building in month $t + 18$. Thus, the *value at time t* of what one obtains immediately at time t upon exercise at time t is the *present value* as of time t of the forward claim, that is, the present value as of time t of a fully operating building at time $t+18$. What is the value of this forward claim?

The present value of a forward claim on an asset is a straightforward DCF problem, equal to the expected value (as of the present) of the future value of the asset (at the time of the forward claim), discounted to present at the risk-adjusted discount rate that reflects the asset's opportunity cost of capital.[14] Thus, for example, in the case of our previously considered office building that is currently worth \$100 million (if it were existing completed new at time 0), the expected value of a similar newly completed building 18 months from now is:

$$E_0[V_{18}] = V_0(1+g_V)^{18} = V_0\left(\frac{(1+r_V)}{(1+y_V)}\right)^{18}$$
$$= \$100(1.0025)^{18} = \$100\left(\frac{(1.0075)}{(1.0050)}\right)^{18} = \$104.57 \textit{ million}$$

And the present value (as of time 0) of the forward claim on this asset is this expectation discounted at the 9 percent/year (or 0.75 percent per month) OCC of office buildings:[15]

$$PV_0[V_{18}] = \frac{E_0[V_{18}]}{(1+r_V)^{18}} = \frac{V_0(1+g_V)^{18}}{(1+r_V)^{18}} = \frac{V_0\left(\frac{(1+r_V)}{(1+y_V)}\right)^{18}}{(1+r_V)^{18}} = \frac{V_0}{(1+y_V)^{18}} \tag{A.4}$$
$$= \frac{\$104.57}{1.0075^{18}} = \frac{\$100}{1.0050^{18}} = \$91.41 \textit{ million}$$

The same is true regarding the construction cost. We have been quoting construction cost as a single lump-sum payment reflecting the entire cost of (instantaneous) construction as of the time when the building is newly complete, and we have assumed a growth trend in these costs of 2 percent per year (simple annual rate, or 0.1652 percent/month) in our example. Assuming a convention of quoting the lump-sum construction cost as of the time when the building is newly complete, the forward construction cost 18 months from now that will be due upon completion of our building, consistent with the similarly defined time 0 construction cost of \$88.24 million that we quoted previously (for a building newly complete at time 0), would be $\$88.24 \times (1.001652)^{18} = \$90.90$ million. The present value (as of time 0) of such a forward construction cost, assuming the OCC for construction cost is the risk-free interest rate of 3 percent per year (or 3%/12 = 0.25% per month), would be: $\$90.90/(1.0025)^{18} = \$86.90$ million. Thus, as of time 0, the relevant underlying asset value for our previously described development option with 18 months time to build is \$91.41 million instead of \$100 million, and the relevant exercise price is \$86.90 million instead of \$88.24 million.

In general, the valuation of the option with time to build is exactly the same as the valuation with instantaneous exercise that we described in section 27A.1.2, except that we replace the current actual values of the underlying asset $V_{i,j}$ and of the construction cost K_j with new values that represent the current values of the forward claims on the underlying asset and the construction cost. Labeling the current (period j) values of these forward claims

[14]Note that a forward claim is not an option; it does not provide the flexibility to forego the claim in an unfavorable outcome. A forward claim is an irrevocable commitment to obtain the asset at the future point in time. Hence, the expected return on an investment in the asset is the OCC appropriate as the discount rate to determine the present value of the claim.

[15]It is somewhat intuitive that the present value of a forward claim on an asset equals its current value discounted by its cash yield rate (y_V), as it is only the cash yield component of the asset's total return that is given up in the interim while the investor does not yet possess the asset. The investor will obtain all of the capital value growth in the asset when she takes possession of the asset as a result of the forward claim.

as $v_{i,j}$ and k_j, respectively, in the option valuation procedure we simply replace $V_{i,j}$ and K_j with:

$$\begin{aligned} v_{i,j} &= V_{i,j}/(1+y_V)^{ttb} \\ k_j &= K_j((1+g_K)/(1+r_f))^{ttb} = K_j\left(\left(\frac{(1+r_f)}{(1+y_K)}\right)/(1+r_f)\right)^{ttb} = K_j/(1+y_K)^{ttb} \end{aligned} \quad \text{(A.5)}$$

respectively (where *ttb* is the time to build measured in number of periods of the binomial model). This replacement is done both in the value trees and in the valuation formulas, and then we proceed as before.

Thus, for example, the option we described previously only now with an 18-month construction time would be worth \$7.48 million, as seen in Exhibit 27A-8, instead of the \$12.57 million we calculated before with instantaneous construction. It makes sense that the development option would be worth less when construction takes 18 months than when construction is instantaneous, as the time to build prevents the developer from obtaining 18 months worth of net rental cash flows that otherwise could be obtained with instantaneous construction.

The general effect of construction time is to reduce the option value, and to increase the currently observable price of similar new buildings that triggers optimal immediate development. This latter effect causes the likelihood of immediate development to fall (and the expected time until development to increase), as a function of the time to build. For example, suppose the ratio $(v_{i,j}/k_j)^*$ is just sufficient to trigger optimal immediate development with time to build, then this corresponds to an observable current value trigger ratio of:

$$\left(\frac{V_{i,j}}{K_j}\right)^* = \left(\frac{1+y_V}{1+y_K}\right)^{ttb}(v_{i,j}/k_j)^* = \left(\frac{(1+r_V)/(1+g_V)}{(1+r_f)/(1+g_K)}\right)^{ttb}(v_{i,j}/k_j)^* \quad \text{(A.6)}$$

As y_V will normally be greater than y_K, the trigger ratio measured in terms of the currently observable values of the underlying asset and construction cost, $(V_{i,j}/K_j)^*$, will be an increasing function of the time to build (*ttb*).[16]

27A.1.4 Risk-Neutral Valuation and the Role of r_V

The option valuation procedure we have described here uses the certainty-equivalence approach to DCF valuation presented in Appendix 10C of Chapter 10. This valuation procedure is applied to a future probability distribution of possible values of the underlying asset and the construction costs (and of the derivative option) in which these probabilities and values represent the actual probabilities and actual possible values, the true dynamics of the relevant underlying asset market.

An alternative methodology is more traditional in much academic literature and is often presented in general finance courses on option valuation theory. This alternative is what is called "risk-neutral valuation," and is based on "risk-neutral dynamics." While the risk-neutral approach is completely equivalent to the approach presented here and gives identical option value solutions, it works with probabilities and future values of the underlying asset that are not "true," but rather what would be consistent within a world in which all investors were "risk-neutral" (that is, they did not care about the risk in their investments).

As investors are obviously not risk-neutral in the real world, the risk-neutral valuation approach may be viewed as simply a mathematical device. However, the nature of this "device," and how it was derived, reminds us of something important about the economics

[16]Normally r_V will be greater than r_f, as the underlying asset is risky, while g_V and g_K will probably be about the same magnitude (approximately equal to the general inflation rate). Thus, y_V will be greater than y_K.

Underlying Asset Value Tree with 18-month Time to Build:
PV of 18-month forward claim on new office building (ex-dividend)

Month ("*j*"): "down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	*n* = 12
0	91.41	96.21	101.26	106.57	112.16	118.05	124.24	130.76	137.63	144.85	152.45	160.45	168.87
1		85.99	90.51	95.26	100.25	105.51	111.05	116.88	123.01	129.47	136.26	143.41	150.93
2			80.90	85.14	89.61	94.31	99.26	104.47	109.95	115.72	121.79	128.18	134.91
3				76.10	80.09	84.30	88.72	93.37	98.27	103.43	108.86	114.57	120.58
					71.59	75.34	79.30	83.46	87.84	92.45	97.30	102.40	107.78
5						67.34	70.88	74.60	78.51	82.63	86.97	91.53	96.33
6							63.35	66.68	70.17	73.86	77.73	81.81	86.10
7								59.60	62.72	66.01	69.48	73.12	76.96
8									56.06	59.00	62.10	65.36	68.79
9										52.74	55.51	58.42	61.48
10											49.61	52.21	54.95
11												46.67	49.12
12													43.90

Construction Cost with 18-month Time to Build:
PV of 18-month forward cost of construction

Month ("*j*"): "down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	*n* = 12
0	86.90	87.04	87.19	87.33	87.48	87.62	87.77	87.91	88.06	88.20	88.35	88.49	88.64
1		87.04	87.19	87.33	87.48	87.62	87.77	87.91	88.06	88.20	88.35	88.49	88.64
2			87.19	87.33	87.48	87.62	87.77	87.91	88.06	88.20	88.35	88.49	88.64
3				87.33	87.48	87.62	87.77	87.91	88.06	88.20	88.35	88.49	88.64
0					87.48	87.62	87.77	87.91	88.06	88.20	88.35	88.49	88.64
5						87.62	87.77	87.91	88.06	88.20	88.35	88.49	88.64
6							87.77	87.91	88.06	88.20	88.35	88.49	88.64
7								87.91	88.06	88.20	88.35	88.49	88.64
8									88.06	88.20	88.35	88.49	88.64
9										88.20	88.35	88.49	88.64
10											88.35	88.49	88.64
11												88.49	88.64
12													88.64

EXHIBIT 27A-8 12-Month Binomial Value Trees with 18-Month Time to Build

American Option Value Tree with 18-month Time to Build:

Month ("j"):	0	1	2	3	4	5	6	7	8	9	10	11	$n = 12$
"down" moves ("i"):	Development possible in any month, takes 18 months from then:												
0	7.48	10.45	14.33	19.24	24.69	30.43	36.48	42.85	49.57	56.65	64.10	71.95	80.23
1		4.45	6.51	9.32	13.07	17.89	23.29	28.97	34.96	41.26	47.91	54.92	62.30
2			2.35	3.63	5.50	8.15	11.78	16.56	21.89	27.52	33.44	39.69	46.27
3				1.04	1.72	2.79	4.44	6.90	10.43	15.23	20.51	26.08	31.94
0					0.35	0.62	1.10	1.92	3.29	5.52	8.97	13.91	19.14
5						0.07	0.13	0.26	0.51	1.00	1.98	3.90	7.69
6							0.00	0.00	0.00	0.00	0.00	0.00	0.00
7								0.00	0.00	0.00	0.00	0.00	0.00
8									0.00	0.00	0.00	0.00	0.00
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

EXHIBIT 27A-8 Continued

that underlie our option valuation procedure (or *any* economically rigorous option valuation procedure), and in so doing provides us with a useful insight that can simplify our use of the model. In particular, the equivalence of risk-neutral and certainty-equivalence valuation of the option implies that the input value we assume for the OCC of the underlying asset, r_V, does not affect the value of the option that we obtain from the model.

Mechanically, the risk-neutral valuation procedure can be defined exactly as we have defined the option valuation procedure here in section 27A.1 only substituting the risk-free interest rate r_f for the underlying asset OCC rate r_V everywhere the latter occurs, that is, both in the computation of the binomial outcome probability p in formula (A.2b) and in the single-period value discounting formula (1) of section 27.3.2 of the main text [which was first introduced in Appendix 10C and appears here in section 27A.1 in formula (A.3b)]. Note that in this substitution in (A.3b) the $E[r_V] - r_f$ term in the numerator becomes $r_f - r_f$, and hence cancels out. This eliminates the "risk discount" component of the certainty-equivalence value in the numerator of (A.3b), leaving the certainty equivalence as just the $E[C_t]$ expectation (which is discounted at the risk-free rate, consistent with a risk-neutral world). However, the modification of the definition of the probability p based on r_f instead of r_V in formula (A.2b) causes $E[C_t]$ now to not be a true expectation (that is, not an expectation in the real world), but rather an expectation in a risk-neutral world. (In such a world, the expected total return on the underlying asset going forward would be simply the risk-free rate.)

The reason why the risk-neutral approach and the certainty-equivalent approach are identical, and why the value of the underlying asset OCC does not impact the valuation of the option, is based on the fundamental economics of how the option valuation procedure works, as described in section 27.3.3 of the main text. The risk-neutral valuation procedure derives directly, mathematically, from the arbitrage valuation approach described in section 27.3.1, which we saw to be equivalent to certainty-equivalence valuation in section 27.3.2, because both are based on the same fundamental cross-market equilibrium condition, as described in section 27.3.3. Thus, *given the value of the underlying asset,* the value of the option is robust to our assumption of what is the opportunity cost of capital (discount rate) for the underlying asset. In effect, given the lock-step relationship between a derivative asset and its underlying asset, everything that we need to know about the underlying asset for purposes of evaluating the derivative is incorporated in the underlying asset's current market value, V_0, given the existence of equilibrium across the relevant asset markets. This is convenient, because it makes our evaluation of the option robust to our (possibly erroneous) assumption about what is the OCC of the underlying asset.

27A.2 A Perpetual Model in Continuous Time

Section 27.5 in the main text presents the Samuelson-McKean model of a perpetual American option as a basic tool to develop intuition regarding land value and optimal development timing. However, the formula presented in the main text is a bit simplified (for example, it assumes a constant, riskless, construction cost), and it does not fully reflect the extent of the Samuelson-McKean model's ability to address the land development option in a realistic way. In this section, we will explore the model in more depth, showing how to increase the realism of the model's assumptions by allowing for construction costs that grow over time, and by allowing for both time and risk in the construction process. We will also present a detailed numerical example of the application of the Samuelson-McKean model.

27A.2.1 Accounting for Time to Build and Risky Construction Costs

The classical Samuelson-McKean formula as presented in section 27.5 of the main text assumes that the option's exercise price is riskless and that option exercise is instantaneous. For the real estate development application, it is more realistic to model the exercise price (construction cost) as possibly having some risk and to consider that construction (the exercise of the option) takes time. The latter fact is often referred to as "time to build." In this section, we will explain how the classical model can be easily adapted to consider these possibilities.

To allow for uncertainty in construction costs, the Samuelson-McKean formula can be subjected to a transformation first proposed by Fisher (1978) and Margrabe (1978) for financial options.[17] In this transformation, the option is evaluated *per dollar of construction cost*. In other words, construction cost is treated as the "numeraire," and K always equals unity. The underlying asset, V, becomes the benefit/cost ratio of building value divided by construction cost. Apart from that, the model is applied as described in the main text, except that the volatility employed in the model is no longer the volatility simply of the underlying asset, but rather that of a portfolio that is long in the underlying asset and short in construction costs. Labeling this volatility σ_P, we have:

$$\sigma_P = \sqrt{\sigma_V^2 + \sigma_K^2 - 2\rho_{VK}\sigma_V\sigma_K} \tag{A.7}$$

Where σ_K is the volatility of construction costs, and $\rho_{V,K}$ is the correlation coefficient between building value and construction costs.

[17] S. Fischer, "Call Option Pricing When Exercise Price Is Uncertain, and Valuation of Index Bonds," *Journal of Finance* 33 (1) (1978): 169–176; W. Margrabe, "Value of an Option to Exchange One Asset for Another," *Journal of Finance* 33 (1) (1978): 177–186.

Accounting for construction time, or "time to build," is also a straightforward and simple modification of the basic Samuelson-McKean formula. We simply replace V_0 and K_0 in the formula with the *present values* of the expected *future values* of the new building and its construction cost, respectively, as of the time of completion of the construction project, after its required time to build. Thus, the main text's formula (2a) for the option elasticity remains unaffected, but letting *ttb* equal the amount of time required for construction, the main text's formula (2b) for the option value becomes:[18]

$$C_0 = (V^* - K_0/\exp(y_K(ttb)))\left(\frac{V_0/\exp(y_V(ttb))}{V^*}\right)^{\eta} \tag{A.2b*}$$

and formula (2c) for the hurdle ratio becomes:

$$V^* = \frac{(K_0/\exp(y_K(ttb))\eta}{\eta - 1} \tag{A.2c*}$$

It is important to note that the hurdle value of the underlying asset, V^*, in this context equals the hurdle *present value* of the expected *future value* that the underlying asset will have after the time to build. Therefore, expressing the hurdle benefit/cost ratio in terms of currently observable values of the underlying asset and the construction cost (as of time 0), we have:

$$(V_0/K_0)^* = \exp((y_V - y_K)(ttb))\frac{\eta}{(\eta - 1)}. \tag{A.8}$$

In other words, assuming that $y_V > y_K$ (as is likely), the greater the time required for construction (larger value of *ttb*), the greater will be the hurdle benefit cost ratio that triggers immediate optimal development, as measured by the current (time 0 observable) values of the underlying asset and its construction cost.

27A.2.2 Numerical Example Application of the Samuelson-McKean Formula

As an example of how we would apply the Samuelson-McKean formula, let's return to the numerical example in section 27.3 of the main text. There we had a land parcel that could be developed today to obtain a new office building that would be worth \$100 million today, with a current construction and development cost exclusive of land totaling \$88.24 million. The development option was valid for only one year, there were only two possible future outcome possibilities next year (either the new building would be worth \$113.21 million or \$78.62 million, with probabilities of 70 percent and 30 percent, respectively, and with construction cost by then of \$90 million in any case), and construction was assumed to be instantaneous. In this situation, the option (the land) was seen to be worth \$12.09 million, and the optimal development timing decision was to wait and hold the land for speculation for possible development next year. The Samuelson-McKean

[18] As the Samuelson-McKean formula works in continuous time, the input rates are "instantaneous rates" (log differences, or continuously compounded rates), as described in section 8.1.6 of Chapter 8. The "ln" and "exp" keys on any business calculator, or the Excel formulas "= ln()" and "= exp()," can be used to compute instantaneous rates and resulting present and future values as described in Chapter 8. Recall that exp(rT)is e^{rT}, referred to as *"rT exponential,"* the base of natural logs raised to the power of the quantity r times T. It may be useful to think of **exp(rT)** as the continuous time equivalent of $\mathbf{(1 + r)}^T$. With this in mind, note that:

$$PV_0[V_{ttb}] - \frac{E_0[V_{ttb}]}{\exp(r_V(ttb))} - \frac{V_0\exp(g_V(ttb))}{\exp(r_V(ttb))} - \frac{V_0\left(\frac{\exp(r_V(ttb))}{\exp(y_V(ttb))}\right)}{\exp(r_V(ttb))} - \frac{V_0}{\exp(y_V(ttb))}.$$

model will allow us to make the development option perpetual, with continuous time and continuous pricing. We can also allow for a realistic time to build, and risky construction costs. The underlying assumption is that returns are normally distributed and uncorrelated across time.

Suppose the annual cash yield provided by office buildings is 6 percent, the construction yield rate is 1 percent (that is, the expected growth rate in construction cost is 1 percent per year less than the OCC of construction costs), and the relevant volatility is 20 percent per year.[19] Suppose further that building the office building will take 18 months (1.5 years). Then, we can apply the Samuelson-McKean formula as follows.

First, the option elasticity is seen to be 3.64, based on formula (2a):

$$\begin{aligned}\eta &= \{y_V - y_K + \sigma_V^2/2 + [(y_K - y_V - \sigma_V^2/2)^2 + 2y_K\sigma_V^2]^{1/2}\}/\sigma_V^2 \\ &= \{.06 - .01 + (.20)^2/2 + [(.01 - .06 - (.20)^2/2)^2 + 2(.01)(.06)^2]^{1/2}\}/(.20)^2 \\ &= 3.64.\end{aligned}$$

This means that an investment in the land will have 3.64 times the return volatility and 3.64 times the investment risk of an investment in a completed office building.

Supposing (as in section 27.3) that the OCC of the completed office building is 9 percent and the risk-free interest rate is 3 percent (implying a risk premium of 6 percent for the office building), then the expected return risk premium in the land investment should be 3.64 times 6 percent, or 21.82 percent. This implies that the OCC for the land, equal to the risk-free rate plus this risk premium, is 24.82% = 3% + 21.82%. It is interesting that this is not terribly different from our original ad hoc assumption of 20 percent for the OCC of the land, back in section 27.2.[20]

Next, we can compute the hurdle benefit/cost ratio based on formula (A.8) above:

$$\begin{aligned}(V_0/K_0)* &= \exp((y_V - y_K)(ttb))\frac{\eta}{(\eta - 1)} \\ &= \exp((.06 - .01)(1.5))\frac{3.64}{(3.64 - 1)} \\ &= \exp(.075)1.38 = 1.4866\end{aligned}$$

This means that as soon as the current value of new office buildings like the one we could build equals 1.4866 times the current construction cost, it will be optimal to immediately commit to construction (which will produce the completed building 18 months later). This implies that, at the time of optimal immediate development, the land value will be 24.02 percent of the expected future value of the completed building at the time of completion of construction.[21]

Thus, as the current construction cost is \$88.24 million, a current value of 1.4866 × \$88.24 = \$131.17 million for the office building would be necessary to trigger immediate

[19] Apart from the noted differences, these inputs are roughly consistent with the example in the main text section 27.3. For example, the 6% cash yield rate could correspond to an expected new building value growth rate of 3% per year and an OCC of 9% on completed building investments ($r_V = g_V + y_V = 3\% + 6\% = 9\%$). However, note that for use in the Samuelson-McKean formula the inputs are instantaneous rates, and hence not exactly comparable to the simple annual rates used in section 27.3. Also, note that the 20% volatility, σ_P, and the construction cost OCC (the construction cost discount rate, r_K), may reflect some risk in construction costs, and some correlation between construction costs and building value, as provided for in formula (A.7) of section 27A.2.1.

[20] However, keep in mind that, unlike in section 27.2, all of the rates in section 27.5 are instantaneous rates. For example, the instantaneous OCC rate of 9% implies a simple annual total return expectation of exp(0.09) − 1 = 9.42%. The option instantaneous OCC of 24.82% equates to a simple annual rate of exp(0.2482) − 1 = 28.18%.

[21] This is based on the present value of the land as a fraction of the future value of the completed building, assuming that development is just optimal now at time 0. (See the numbers in the following footnote.)

optimal development.[22] As the current actual value of the office building we could develop is only \$100 million, it is optimal to wait and hold the land undeveloped for now.

Equivalently, we could compute the present value of the future completed building (after the required construction time of 1.5 years) that would make immediate development optimal. This value is given by formula (2c*) as \$119.88 million:

$$V^* = (K_0/\exp(y_K(ttb)))\frac{\eta}{\eta - 1}$$
$$= (\$88.24/\exp(.01(1.5)))\frac{3.64}{3.64 - 1} = (\$88.24/1.0151)\frac{3.64}{2.64} = \$119.88$$

As the actual present value of a forward claim on an office building in 1.5 years is only \$91.39 million [computed as (\$100)exp[(0.09 − 0.06)1.5]/exp[(0.09)1.5] = \$100/exp[(0.06) 1.5] = \$100/1.0942 = \$91.39], we do not currently have sufficient value in the underlying asset to justify immediate development.

Finally, we can compute the current value of the land, based on formula (2b*), as follows:

$$C_0 = (V^* - K_0/\exp(y_K(ttb)))\left(\frac{V_0/\exp(y_V(ttb))}{V^*}\right)^{\eta}$$
$$= (119.88 - 88.24/1.0151)\left(\frac{91.39}{119.88}\right)^{3.64} = \$32.96(0.7624)^{3.64} = \$12.28$$

Thus, the land is currently worth \$12.28 million.

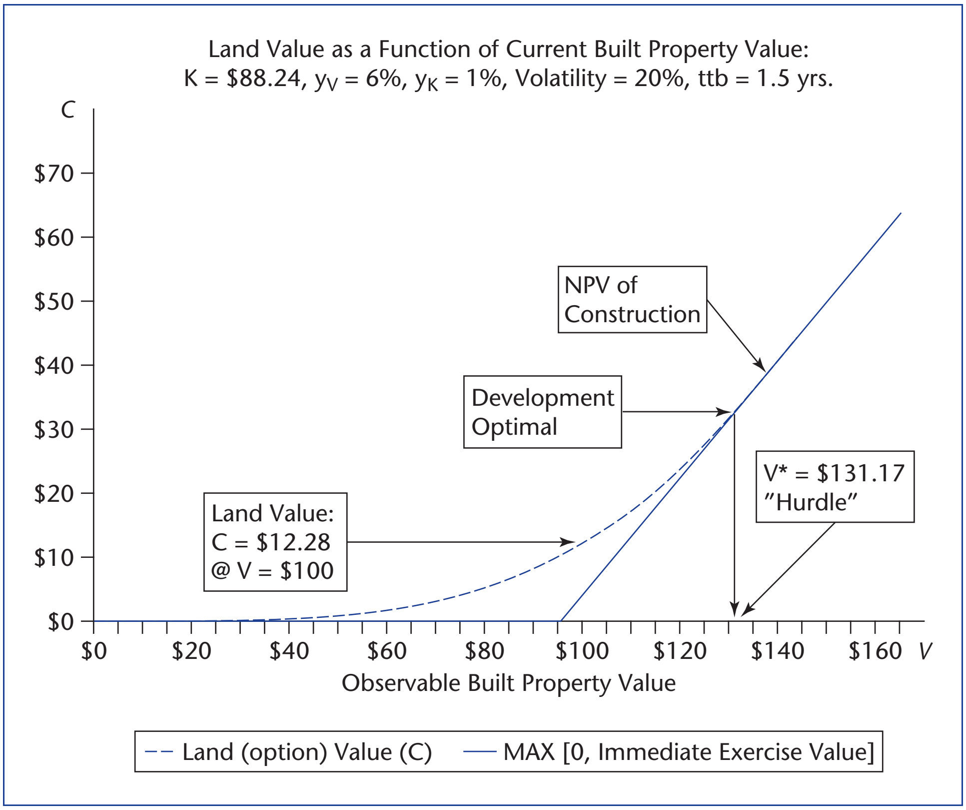
It is interesting that this value is not much different from the \$12.09 million that we computed with our very simple one-year binomial example back in section 27.3. This similarity is due to two offsetting effects. On the one hand, the perpetual option is worth considerably more than the one-year option, holding everything else the same. In particular, if we could have instantaneous construction, the Samuelson-McKean valuation of the land would be \$16.38 million (instead of \$12.28 million, and the current hurdle value of the office building would be \$121.69 million instead of \$131.17 million). On the other hand, recognizing the

[22] This immediate development situation would correspond to an expected completed building value in 18 months of: (\$131.17)exp[(.03)1.5] = \$137.21 million. If the construction OCC is 3%, then the expected construction cost as of the time of project completion would then be: (\$88.24)exp[(.03 − .01)1.5] = \$90.92 million, for an anticipated construction profit of \$137.21 − \$90.92 = \$46.28 million on completion (exclusive of land cost). In present value terms (at "time 0," when the development commitment would be made, 1.5 years before completion), this would provide a residual of \$137.21/exp[(.09)1.5] − \$90.92/exp[(.03)1.5] = \$131.17/exp[(.06)1.5] − \$88.24/exp[(.01)1.5] = \$119.88 − \$86.92 = \$32.96 million, which would be the present value of the land today if we were at the immediate optimal development situation (new office building worth \$131.17 million today). This would provide an (instantaneous) expected annual return rate on the development project of ln(46.28/32.96)/1.5 = 22.64%, slightly less than the 24.82% OCC rate for the land speculation (if continued speculation were optimal, which it wouldn't be in this case). The committed development project expected return is less than that of an optimally "live option" (being held unexercised), because the latter is more risky (effectively more "leveraged") than a committed development project. As will be discussed in more depth in Chapter 29, the 22.64% expected return on the committed development project is entirely consistent with the cross-market equilibrium asset pricing framework described in section 27.3.3 of this chapter. That is:

$$\frac{E_0[V_{ttb}] - E_0[K_{ttb}]}{\exp((r_C)ttb)} - \frac{E_0[V_{ttb}]}{\exp((r_V)ttb)} - \frac{E_0[K_{ttb}]}{\exp((r_K)ttb)}$$
$$\Rightarrow \frac{\$137.21 - \$90.92}{\exp((.2264)1.5)} - \frac{\$137.21}{\exp((.09)1.5)} - \frac{\$90.92}{\exp((.03)1.5)}$$

where: V is an asset or claim that is traded in the built property market and has OCC of r_V characteristic of that market; K is an asset or claim that has contractually fixed cash flows (like a riskless bond) and has OCC of r_K characteristic of the risk-free bond market; and $V - K$ is the profit (conceivably negative ex post) from a committed development project, a claim that is traded in the market for development projects, whose OCC is (therefore) r_C. The equation provides for equal expected return risk premium per unit of risk, across the three types of investment markets.

EXHIBIT 27A-9
Samuelson-McKean Model Land Value as a Function of Current Built Property Value

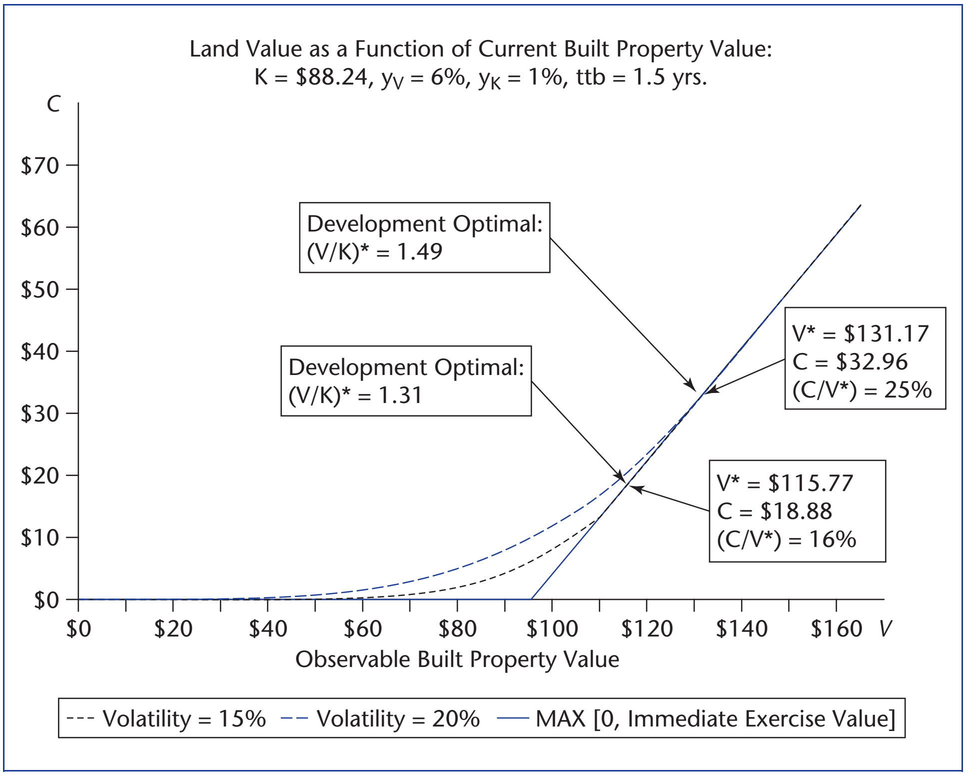


1.5 year time required for construction reduces the value of the option. If we had applied a 1.5 year time to build within the framework of the one-period binomial model of section 27.3, we would have computed a value of only $7.83 million (instead of $12.09 million).

Exhibit 27A-9 displays graphically the numerical example we have just presented. The horizontal axis represents the currently observable value of the underlying asset, V_0, and the vertical axis the present value of the land (or of the development project). The straight line coming out of the horizontal axis at $V_0 = \$95.11$ million gives the current NPV of a commitment to proceed with development (excluding consideration of the land cost). This value is simply the present value of the forward claim on the new building minus the present value of its construction cost: $V_0/\exp[y_V(ttb)] - K_0/\exp[y_K(ttb)]$, which equals zero at $V_0 = \$95.11$ when $K_0 = \$88.24$, $y_V = 6\%$, $y_K = 1\%$, and $ttb = 1.5$ years. Notice that the Samuelson-McKean land value, given by the upward-sloping convex curved line, exceeds this construction NPV until the point where on the horizontal axis V_0 equals the hurdle value of $131.17 million, at which point the land (option) is worth $32.96 million, exactly equal to the construction NPV: $V_0/\exp[y_V(ttb)] - K_0/\exp[y_K(ttb)] = \$131.17/\exp[0.06(1.5)] - \$88.24/\exp[0.01(1.5)] = \$119.88 - \$86.92 = \$32.96$ million. At current building values less than $131.17 million, the *complete* NPV of the development project, including subtracting the opportunity cost of the land (the option given up), would be negative.

Exhibit 27A-10 shows how the option value is affected by a reduction in the relevant volatility. (A similar effect results from an increase in the built property yield rate, y_V, or from a decrease in the construction yield rate y_K.) Note that the option value is reduced for all values of the built property, and the hurdle value at which development is optimal is also

EXHIBIT 27A-10
Samuelson-McKean Model Land Value as a Function of Current Built Property Value



reduced. A reduction in relevant volatility from 20 percent to 15 percent reduces the hurdle benefit/cost ratio from 1.49 to 1.31 and reduces the land value fraction of total property value at the moment of optimal development from 25 percent to 16 percent of the current property value.[23]

[23] V^* in Exhibit 27A-10 is measured in terms of the currently observable built property value, rather than as in formula (2c*)in terms of the present value of the future building that could be obtained from the development. For example, with 20% volatility, at the point of optimal immediate development, the land value of $32.96 million equals: (i) 25.1% of the $131.17 million current observable built property value [V_0]; or (ii) 24.0% of the $137.21 million projected future building value on completion {which equals $V_0\exp(g_V(ttb)) = 131.17\exp[0.03(1.5)]$}; or (iii) 27.5% of the $119.88 million present value of the projected completed building value {which equals $E_0[V_{ttb}]/\exp[r_V(ttb)] = \$137.205/\exp[0.09(1.5)]$}. The land value fraction at the point of optimal development defined in this last way (as a fraction of the present value of the future completed building) simply equals $1/\eta$, the inverse of the option elasticity [as $1/3.64 = 27.5\%$], the same as it would be if ttb = 0.